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Exact results for car accidents in a traffic model

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Abstract. Within the framework of a recent model for car accidents on single-lane highway traffic, we study analytically the probability of the occurrence of car accidents. Exact results are obtained. Various scaling behaviours are observed. The linear dependence of the occurrence of car accidents on density is understood as the dominance of a single velocity in the distribution.

Recently, traffic problems have attracted considerable attention [1]. Among the various approaches applied to describe the traffic flow, the cellular automata models are capable of reproducing many characteristic features. The basic model consists of N cars moving on a one-dimensional lattice of L sites with periodic boundary conditions. Each site can be either empty, or occupied by exactly one car. All cars can only move in one direction. The velocity of each car is an integer between 0 and v, where v is the speed limit and is often taken to be 5 [2]. The position of the ith car at time t is denoted by $x(i,t)$. The position of the car ahead at time t is then $x(i+1,t)$. The system evolves according to the following synchronous rule

$$x(i,t+1) = x(i,t) + \min\{x(i+1,t) - x(i,t) - 1,\ x(i,t) - x(i,t-1) + a,\ v\}. \tag{1}$$

The three terms $x(i+1,t) - x(i,t) - 1$, $x(i,t) - x(i,t-1) + a$ and v represent the driving schemes respecting the safety distance, acceleration a and the speed limit, respectively. The case of $a = 1$ corresponds to a well known model [2], while the case of $a = v$ can be considered as a variation for aggressive driving [3]. In this last case, the evolution rule can be written as

$$x(i,t+1) = x(i,t) + \min\{x(i+1,t) - x(i,t) - 1,\ v\}. \tag{2}$$

More recently, the occurrence of car accidents has been studied numerically within this framework [4]. Car accidents occur when drivers do not respect the safety distance, which often happens when the car ahead is moving. At time t, if the velocity of car $i+1$ is positive, expecting this velocity to remain positive at time $t+1$, the driver of car i increases the safety velocity by one unit, with probability p. The safety distance $x(i+1,t) - x(i,t) - 1$ in equation (2) is replaced by $x(i+1,t) - x(i,t)$, with probability p. If the car $i+1$ stops at time $t+1$, this careless driving will result in an accident. The probability *per car* and *per time step* for an accident to occur is denoted by P_{ac}. In [4], the value of P_{ac} as a function of car density $\rho = N/L$ is studied numerically for a special case $v = 3$. When the car density is less than the critical value $\rho_{\mathrm{c}} = (1+v)^{-1}$, no accident occurs. The average velocity is v and the average distance between two consecutive cars is larger than v. The fraction of stationary cars, which is denoted by n_0, is zero. When $\rho > \rho_{\mathrm{c}}$, the average

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velocity decreases with ρ, the average distance decreases with ρ, the fraction n_0 increases with ρ, and careless driving will result in a number of accidents. In the case of $\rho = 1$, all cars are stopped and no accident occurs. The probability for a car accident to occur is observed, as expected, to reach a maximum for a car density between ρ_c and 1.

In the following, we present the exact results for these numerical features. Exact results are obtained for the case $a = v$, i.e. the evolution rule of equation (2). The explicit expression for the distribution $P_{ac}(\rho)$ is also given.

For a careless driver to have an accident, the necessary conditions are such that the next car ahead is moving at time t and then stopped at time $t+1$. These conditions at different time steps can be replaced by conditions at the same time step. The car ahead stops at time $t+1$ because that two cars ahead stops at time t. Thus the criteria for an accident to occur can be rewritten as four constraints at the same time step: at time t, (1) the velocity of the car ahead is positive, (2) the velocity of the second car ahead is zero, (3) the number of empty sites to the car ahead is less than v, and (4) the number of empty sites between the car ahead and that two cars ahead is zero. With these four constraints satisfied, an accident is expected to occur with probability p. Obviously, P_{ac} is proportional to p.

For the model without car accidents, a phenomenological mean-field theory turns out to be exact [3]. The same results can also be obtained by constructing the microscopic Boolean variable [5]. (In the case of $a = 1$, the mean-field theory only provides an approximate result [6].)

Let d be the probability for a site being empty. The fractions of cars with various velocities can be related to the probability d as

$$n_i = \begin{cases} d^i(1-d) & \text{for } 0 \leqslant i < v \\ d^i & \text{for } i = v \end{cases} \tag{3}$$

where n_i denotes the fraction of cars with velocity i and the effects of correlations have been included. With the known results of the car flow, an equation for d can be obtained

$$\sum_{i=1}^{v} d^i = \frac{1-\rho}{\rho}. \tag{4}$$

In principle, the ρ-dependence of probability d, and thus n_i, can be determined explicitly.

In [4], neglecting correlations, the probability d was taken as $1-\rho$. With a further assumption on n_0, a crude approximation was obtained for P_{ac}. The analytical expression for the probability P_{ac} can be obtained within the mean-field theory, where the value of P_{ac} is written as

$$P_{ac} = \frac{p}{\rho}\left(\sum_{i=1}^{v} \rho\ n_i\right)(\rho n_0)\left(\sum_{n=0}^{v-1} d^n\right). \tag{5}$$

The summation over i results in the probability for a site being occupied by a car with positive velocity, which provides the constraint on the velocity of the car ahead. The term ρn_0 is the probability for a site being occupied by a car with zero velocity, which then provides the constraint on the velocity of the car two cars ahead. The summation over n takes into account the constraint on the number of empty sites to the car ahead. With a factor d^0 (=1) assumed, the number of empty sites between the car ahead and that two cars ahead, is zero.

With the help of equation (4), the probability P_{ac} can be reduced to

$$P_{ac} = p(1-\rho)(1-d). \tag{6}$$

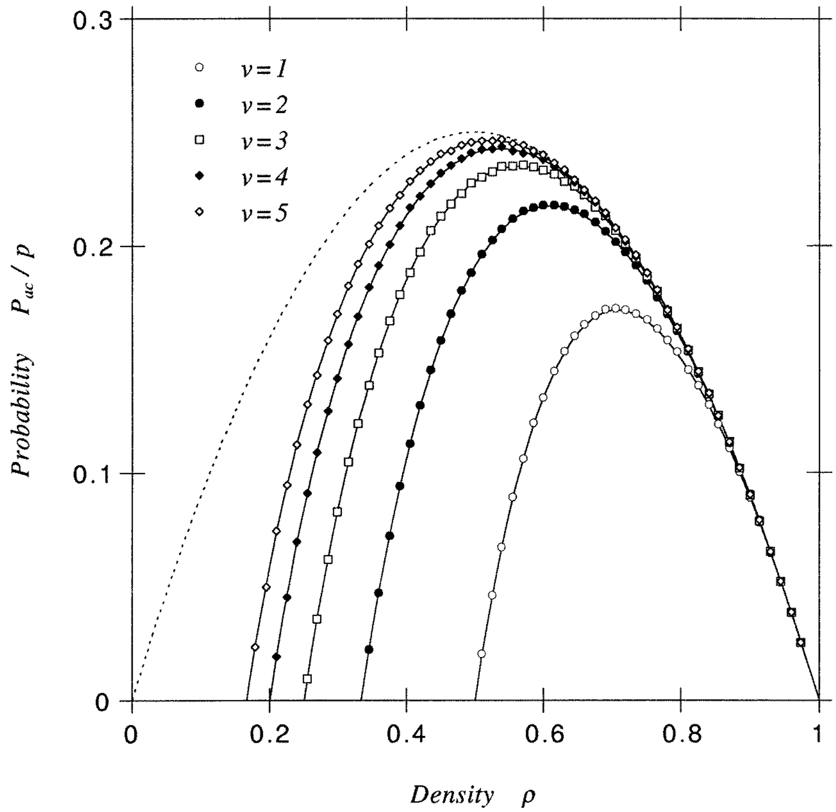

Figure 1. Probability P_{ac}/p as a function of density ρ, for various speed limits v. The various symbols are numerical results for $v = 1, 2, \ldots, 5$. The full curves are the corresponding results from equation (7). The dotted curve is the result of the limit $v \to \infty$, equation (10).

The relation between P_{ac} and ρ can thus be established

$$\sum_{i=1}^{v}\left[1 - \frac{P_{ac}}{p(1-\rho)}\right]^i = \frac{1-\rho}{\rho}. \tag{7}$$

In principle, the ρ-dependence of probability P_{ac} can be determined explicitly. The results are shown in figure 1.

In the case of $v = 1$, we have

$$P_{ac} = \begin{cases} 0 & \text{for } \rho < \frac{1}{2} \\ p\left(2 - \dfrac{1}{\rho}\right)(1-\rho) & \text{for } \rho \geqslant \frac{1}{2}. \end{cases} \tag{8}$$

As the density ρ increases, the onset of the probability P_{ac} occurs at a density $\rho = \frac{1}{2}$ and reaches its maximum $(3 - 2\sqrt{2})p$ at a density $\rho = \frac{1}{\sqrt{2}}$.

In the case of $v = 2$, we have

$$P_{ac} = \begin{cases} 0 & \text{for } \rho < \frac{1}{3} \\ p\left(\dfrac{3}{2} - \sqrt{\dfrac{1}{\rho} - \dfrac{3}{4}}\right)(1-\rho) & \text{for } \rho \geqslant \frac{1}{3}. \end{cases} \tag{9}$$

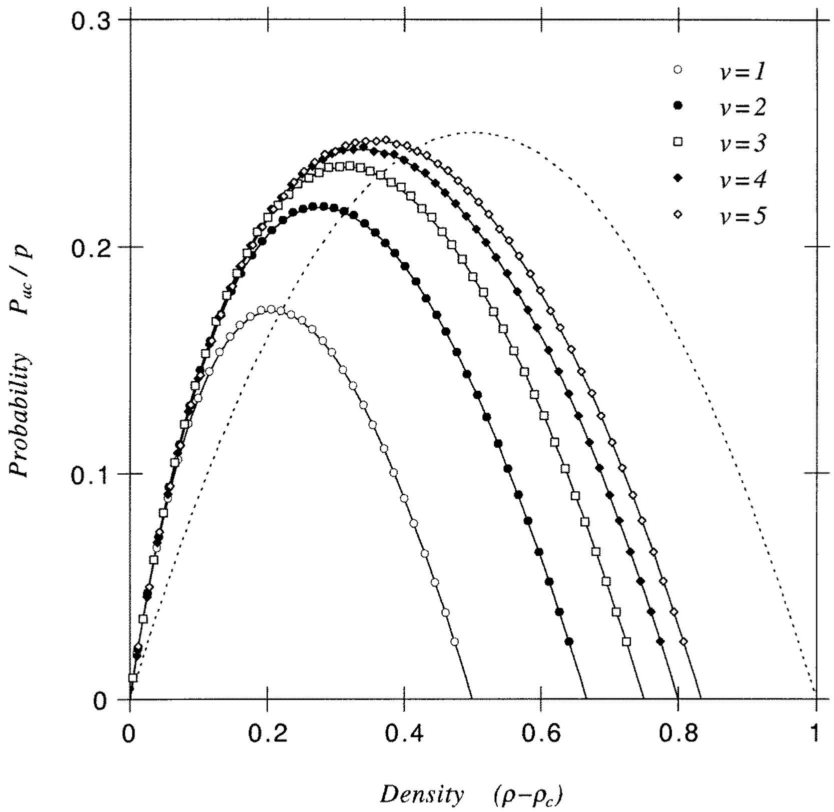

Figure 2. Probability P_{ac}/p as a function of the shifted density $(\rho - \rho_c)$. The symbols are the same as in figure 1.

The onset of the probability P_{ac} occurs at a density $\rho = \frac{1}{3}$ and reaches its maximum $0.217\,211p$ at a density $\rho = 0.609\,674$; (exact expressions for these numbers can also be found, which are not shown).

As the speed limit v increases, the value of P_{ac} also increases. The onset of P_{ac} reduces to a density $\rho = (1+v)^{-1}$ and the density for the maximum P_{ac} also shifts to a lower value. In the limit $v \to \infty$, i.e. without imposing a speed limit, the probability for an accident to occur becomes

$$P_{ac} = p\rho(1-\rho). \tag{10}$$

The value of P_{ac} has a maximum $\frac{1}{4}\ p$ at a density $\rho = \frac{1}{2}$, see figure 1.

With the analytical expression of equation (7), various properties for the occurrence of car accidents can be studied in detail. Around the critical density $\rho_c = (1+v)^{-1}$, the onset of probability P_{ac} is linear,

$$P_{ac} \sim 2p(\rho - \rho_c). \tag{11}$$

It is interesting to note that a universal value of the slope is observed, see figure 2. The slope of the rise has a value of $2p$, which is independent of the speed limit v. However, in the limiting case, $v \to \infty$, the slope becomes p, see equation (10) and figure 2. It is an indication that the simple result of equation (10) is not trivial and cannot be obtained perturbatively.

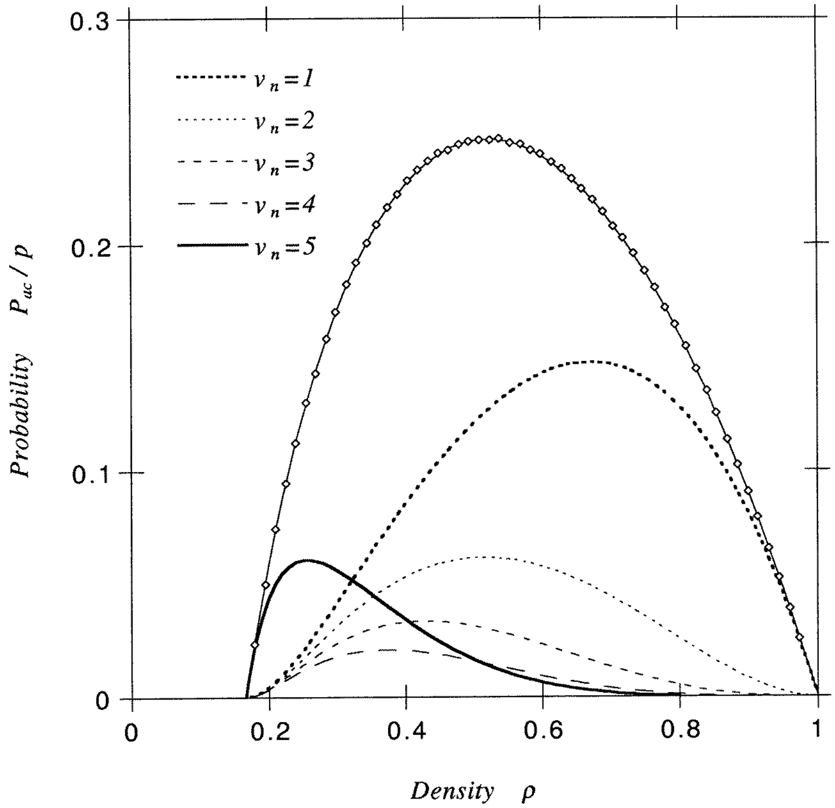

Figure 3. Probability P_{ac}/p as a function of density ρ, for various speed v_n. (v_n is the velocity of the car ahead of the careless driver.) The symbols and full curve are the same as in figure 1 for $v = 5$.

At very high density, $\rho \sim 1$, we observe another linear behaviour between P_{ac} and ρ,

$$P_{ac} \sim p(1-\rho) \tag{12}$$

which is also universal, see figure 1. A linear drop is noticed with a universal slope at a value of $-p$.

Next we study the velocity distribution of cars which resulted in an accident. Rewriting equation (5), the probability P_{ac} can be decomposed into contributions from various velocities,

$$P_{ac} = \sum_{i=1}^{v} P_{ac,i} \tag{13}$$

$$= \sum_{i=1}^{v} \left[\frac{p}{\rho} (\rho n_i)(\rho n_0) \left(\sum_{n=0}^{v-1} d^n \right) \right] \tag{14}$$

where the index i denotes the velocity of the car being hit, i.e. the car ahead to the careless driver. The results are shown in figure 3 (the index i is denoted by v_n in the figure). Around the critical density ρ_c, most of the cars being hit from behind drive with the maximum velocity v. As the density increases, the accidents resulting from cars driving with slower velocities increases. The rate of increase is the largest for the slowest cars, $v_n = 1$, and

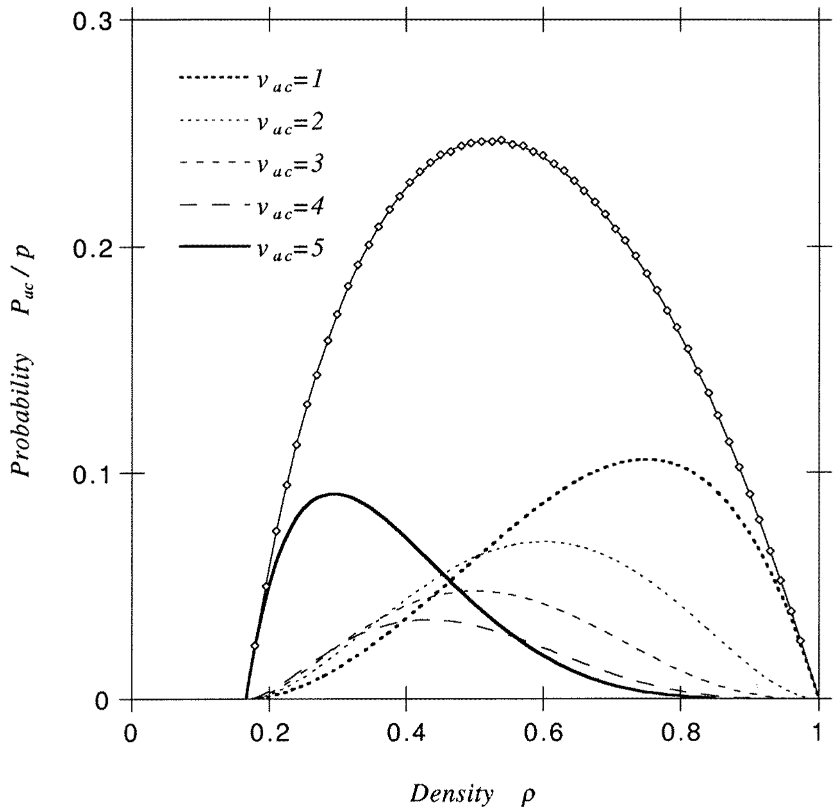

Figure 4. Probability P_{ac}/p as a function of density ρ, for various speed v_{ac}. (v_{ac} is the velocity of the careless driver.) The symbols and full curve are the same as in figure 1 for $v = 5$.

decreases with the increase of velocity. We note that stationary cars will not be hit in this model. Around the highest density, $\rho = 1$, most of the cars being hit drive with the lowest velocity, $v_n = 1$.

The probability P_{ac} can also be decomposed into contributions from various velocities of the careless driver (which is denoted by v_{ac} in the figure.) The results are shown in figure 4. Similar features are also observed. The linear behaviours around the densities ρ_{c} and 1 can be understood as the result of one velocity dominating over others. Around the critical density ρ_{c}, most of the accidents result from cars driving with maximum velocity v, for both v_{ac} and v_n; while close to $\rho = 1$, the dominant velocity becomes that with the lowest value, i.e. $v_{\mathrm{ac}} = v_n = 1$.

In summary, we study analytically the occurrence of car accidents in a traffic model. The exact results are obtained from the phenomenological mean-field theory. The explicit expression for the probability of the occurrence of car accidents is presented. The universal scaling behaviour and the distribution of velocities in accidents are studied.

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